EXERCISE – IV HINTS & SOLUTIONS

Sol.1 Let (h, k) is middle point of coc of hyperbola.
so its equation is $T = S_1$

$$\Rightarrow \frac{xh}{a^2} - \frac{yk}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad ......(i)$$

But also this equation has arisen due to a point P($r\cos\theta$, $r\sin\theta$) on the circle
so equation of this line must be
$x\cos\theta + y\sin\theta = r$ ......(ii)
comparing equation (i) & (ii)

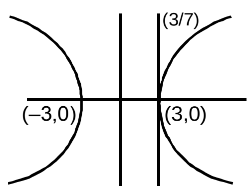
$$\frac{h/a^2}{\cos\theta} = -\frac{k/b^2}{\sin\theta} = \frac{h^2/a^2 - k^2/b^2}{r}$$

$$\Rightarrow \cos\theta = \frac{h.r}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right) \ \& \ \sin\theta = \frac{-kr}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)$$

on eliminating θ the locus of (h, k) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = \left(\frac{x^2+y^2}{r^2}\right)$$

Sol.2 Equation tangents

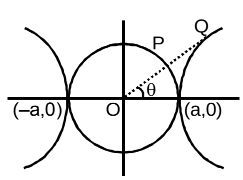


using $y = mx \pm \sqrt{9m^2 - 1}$
$\downarrow$
$m = \infty$ & $m = 5/12$

so $x = 3$ & $y = \frac{5}{12}x + \frac{3}{4}$

equation of COC ; $3x - 18y - 9 = 0$
so area = 8 sq. units

Sol.3 Point P on the circle is ($a\cos\theta$, $a\sin\theta$)
Let $y = mx$ is the line which cuts the hyperbola at θ (where $m = \tan\theta$)



$$\text{then } x = \frac{a}{\sqrt{1-m^2}} = \frac{a}{\sqrt{1-\tan^2\theta}} \ \& \ y = \frac{a\tan\theta}{\sqrt{1-\tan^2\theta}}$$

$$\text{so point } \theta = \left(\frac{a}{\sqrt{1-\tan^2\theta}}, \frac{a\tan\theta}{\sqrt{1-\tan^2\theta}}\right)$$

Tangent at P $\Rightarrow$ $x\cos\theta + y\sin\theta = a$ .....(i)
Tangent at θ $\Rightarrow$ $x\cos q - y\sin\theta = a\cos\theta\sqrt{1-\tan^2\theta}$ .....(ii)
Let the Intersection point of (i) & (ii) is (h, k)

$$\text{then } h = \frac{a(\cos\theta\sqrt{1-\tan^2\theta} + 1)}{2\cos\theta} \quad .....(iii)$$

$$\& \ k = \frac{-a\sqrt{1-\tan^2\theta}}{\tan\theta} \quad .....(iv)$$

Using (iii) & (iv) locus of (h, k) is
$a^4(x^2 - a^2) + 4x^2y^4 = 0$.

Sol.4 Let the end of the latus rectum is (ae, b^2/a)

& Let the equation of asymptotc is $\frac{x}{a} + \frac{y}{b} = 0$ ...(i)

so equation of normal, parallel to (i) can be written as :

$y = mx + \sqrt{a^2m^2 - b^2}$ & $m = -b/a$

$$\Rightarrow y = -\frac{b}{a}x + \sqrt{b^2 - b^2}$$

$$\Rightarrow y = -\frac{b}{a}x \quad ......(ii)$$

$\downarrow$ passes through (ae, b^2/a)

$$\frac{b^2}{a} = -\frac{b}{a}ae \Rightarrow \frac{b}{a} = -e$$

so eccantricity $e = \frac{1+\sqrt{5}}{2}$.

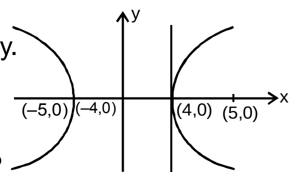
Sol.5 Equation of tangent at (9, 4)
$x - y - 5 = 0$

solving with $y = \frac{2}{3}x$ & $y = -\frac{2}{3}x$

point Q & R are (15, 10) and (3, –2)
Also P is the middle point of QR.

$$\text{Area of } \Delta CQR = \left|\frac{1}{2}\begin{vmatrix} 15 & 10 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{vmatrix}\right| = 30 \text{ sq. units.}$$

Sol.6 S & S' are (5, 10) & (–5, 0) respectively.
point P = (–1, 0)
equation of line through P with 135º with axis OX is



$x + y + 1 = 0$ ......(i)

Its intersection with $y = 3/4\ x$ & $y = -\ 3/4\ x$ is $(-4, 3)$ & $(-4/7, -3/7)$

Sol.7 Asymptotc of hyberbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ is

$$y = \frac{3}{4}x \qquad ....(1)$$

equation of diameter :

$$y = -\frac{4}{3}x \qquad ....(2)$$

solving with ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

point of intersection = $\left(\dfrac{45}{\sqrt{481}}, \dfrac{-60}{\sqrt{481}}\right)$

& $\left(\dfrac{-45}{\sqrt{481}}, \dfrac{60}{\sqrt{481}}\right)$

$$\ell_{\text{diameter}} = \sqrt{\frac{(90)^2}{(\sqrt{481})^2} + \frac{(120)^2}{(\sqrt{481})^2}} = \frac{150}{\sqrt{481}}$$

Sol.8 Let the equation of the tangent at $(a \sec\theta, b\tan\theta)$

bx sec q – ay tan q = ab ....(i)

equation of the asymptote, $y = \dfrac{b}{a}x$ ....(ii)

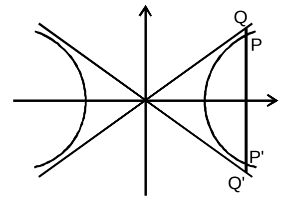
solving (i) & (ii) point $\theta = (a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta))$

Let the mid point is (h, k) then using mid point formula

so $\dfrac{h}{a} = \sec\theta + \dfrac{\tan\theta}{2}$ & $\dfrac{k}{b} = \dfrac{\sec\theta}{2} + \tan\theta$

$\Rightarrow$ locus of (h, k) is a similar hyperbola

Sol.9 Let the line is
$y = mx + c$ ...(i)
solve the equation of H.B. with line



$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0 \text{ ...(3)} < {x_1 \atop x_2}$$

solve the equation of asymptotes with line (i)

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2c^2 = 0 \text{ ...(4)} < {x_3 \atop x_4}$$

from equation (3) & (4) sum of roots is same, hence the mid point of PP′ & QQ′ is same.
so PQ = P′Q′ & PQ′ = P′Q (H.P.)

Sol.10 Let the point on $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

is $(a\sec\theta, b\tan\theta)$

$\Rightarrow -bs\sec\theta + ay\tan\theta - ab = 0$ .....(i)

Now equation of asymptotoes are

$y = \dfrac{b}{a}x$ & $y = -\dfrac{b}{a}x$

solving with COC, point of intersection are

$[-a(\tan\theta + \sec\theta), -b(\tan\theta + \sec\theta)]$ &

$[-a(\sec\theta - \tan\theta), b(\sec\theta - \tan\theta)]$

Now area of Δ formed

$$A = \left|\frac{1}{2}\begin{vmatrix} -a(\tan\theta + \sec\theta) & -b(\tan\theta + \sec\theta) & 1 \\ a(\tan\theta - \sec\theta) & -b(\tan\theta - \sec\theta) & 1 \\ 0 & 0 & 1 \end{vmatrix}\right|$$

= ab (which is constant)

Sol.11 $\dfrac{x - a\sec\theta}{\cos\phi} = \dfrac{y - b\tan\theta}{\sin\phi} = r$

for $r = r_1$ point Q

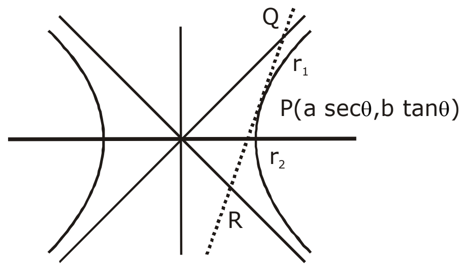
$r = r_2$ point R

$Q[r_1\cos\phi + a\sec\theta, r_1\sin\phi + b\tan\theta]$

$R[r_2\cos\phi + a\sec\theta, r_2\sin\phi + b\tan\theta]$

Q Point lie on $y = \dfrac{b}{a}x$

$$\Rightarrow r_1 = \frac{ab(\sec\theta - \tan\theta)}{a\sin\phi - b\cos\phi}$$



$\tan\phi = m$(given)

R point lie on $y = -\frac{b}{a}x$

$$r_2 = \frac{-ab(\sec\theta + \tan\theta)}{a\sin\varphi + b\cos\varphi}$$

$$r_1r_2 = \frac{-a^2b^2(\sec^2\varphi)}{a^2\tan^2\varphi - b^2}$$

$$= \frac{a^2b^2(1+\tan^2\varphi)}{b^2 - a^2\tan^2\varphi} = \frac{a^2b^2(1+\tan^2)}{b^2 - a^2m^2}$$

Sol.12 $A(t_1, c/t_1)$ & $B(ct_2, c/t_2)$ be extrimities of chord AB & P(h, k) is mid point.

than $t_1 + t_2 = \frac{2h}{c}$ & $\frac{t_1+t_2}{t_1t_2} = \frac{2k}{c}$

$\Rightarrow\ t_1 + t_2 = \frac{2h}{c},\ t_1t_2 = \frac{h}{k}$ .....(i)

also $4d^2 = (ct_1 - ct_2)^2 + (c/t_1 - c/t_2)^2$

$$= c^2[(t_1+t_2)^2 - 4t_1t_2]\left(\frac{1+t_1^2\,t_2^2}{t_1^2\,t_2^2}\right)$$

using (i)

$$\Rightarrow 4d^2 = c^2\left(\frac{4h^2}{c^2} - \frac{4h}{k}\right)\left(\frac{1+h^2/k^2}{h^2/k^2}\right)$$

$\Rightarrow\ (h^2 + k^2)(hk - c^2) = d^2hk$

so locus $\Rightarrow\ (x^2 + y^2)(xy - c^2) = d^2xy$

Sol.13 Since the sides of Δ touches $y^2 = 4ax$ so

Let their equation are

$y = m_1x + a/m_1$, $y = m_2x + a/m_2$ & $y = m_3x + a/m_3$

$$\Rightarrow\ \text{vertices of } \Delta = \left(\frac{a}{m_1m_2}, a\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right),$$

$$\&\ \left(\frac{a}{m_2m_3}, a\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right)$$

Let two of the above vertices lie on $xy = c^2$ then we have to prove that remaining one also lies on the curve for vertex.

$$\left(\frac{a}{m_1m_2}\right)a\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = c^2 \quad .....(i)$$

$$\text{Similarly } a^2\left(\frac{1}{m_2} + \frac{1}{m_3}\right) = c^2(m_2\,m_3) \quad .....(ii)$$

subtracting (i) & (ii) $\Rightarrow\ a^2 = -c^2 m_1 m_2 m_3$

$$\Rightarrow\ m_2 = -\frac{a^2}{c^2\,m_1m_3} \quad .....(iii)$$

$$\text{using (i) } \frac{a^2}{m_1} - c^2 m_1 m_3 = \frac{-a^2}{m_3}$$

$$\Rightarrow\ a^2\left(\frac{1}{m_1} + \frac{1}{m_3}\right) = c^2 m_1 m_3$$

(similar to equation (i) & (ii))

which shows that 3rd vertex also lie on the curve.

Sol.14 equation of normal at (ct, c/t)

It passes through A (ct′, c/t′)

It passes through A(ct′, c/t′)

$\Rightarrow\ ct'\,t^4 - ct'\,t^3 - ct + ct' = 0$

If should have four roots t_1, t_2, t_3, t_4, where t_1, t_2, t_3 correspond to P, Q & R while ty to A.

$\Rightarrow\ t_1 + t_2 + t_3 + t_4 = t',\ 2t_1t_2 = 0$

& ty = t' $\Rightarrow\ t_1 + t_2 + t_3 = 0$

$\Rightarrow$ In $\Delta t_1t_2 = t_1t_2 + t_2t_3 + t_3t_1 = 0$

$$\text{Now controid of } \Delta PQR = \left(\frac{c(t_1+t_2+t_3)}{3}, \frac{c\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\right)}{3}\right)$$

$$= \left(0, \frac{c}{3}(t_1t_2 + t_2t_3 + t_3t_1)\right)$$

= (0, 0)

which is the centre of the curve.